Real Tornado

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Abstract

The continued fraction expansion of a real number $R > 0$ generates a family of spiral triangular patterns, called "tornadoes." Each tornado consists of similar triangles, any two of which are non-congruent.

Basic Operation

Let $R > 0$ and $0 < s < 1$. In the plane, the sequence of points $V(j) = (s^j \cos 2\pi jR, s^j \sin 2\pi jR)$ for $j = 0,1,\cdots$, which we call the 'vertices', naturally converges to the origin. Fix an integer $k > 0$, which is called the 'modulo' or the 'step size', and join the vertex $V(j)$ with $V(j+k)$ by the line segment $\overline{V(j)V(j+k)}$ for $j \geq 0$.

Fibonacci Tornado

The Fibonacci numbers f_n are defined by $f_1 = f_2 = 1$ and $f_n = f_{n-2} + f_{n-1}$, $n > 2$. In the previous paper [2], we showed that if $k = f_{n-1}$ and $R = \tau$, where $\tau = (1+\sqrt{5})/2$ is the golden ratio, there exists a $0 < s < 1$ such that the vertex $V(j + f_{n+2})$ lands on the line segment $\overline{V(j + f_{n+1})V(j + f_n)}$ for each $j \geq 0$. By the Basic Operation above, we obtain the spiral pattern of similar triangles as shown in Figure 1 ($k = 2$), which is called a "tornado". As k gets larger, we could see that the tornado comes out like a blooming flower, while the argument jR of each vertex $V(j)$ remains unchanged.

Remark that the well-known spirals as in Figure 2 are different from our tornadoes because they have congruent triangles.

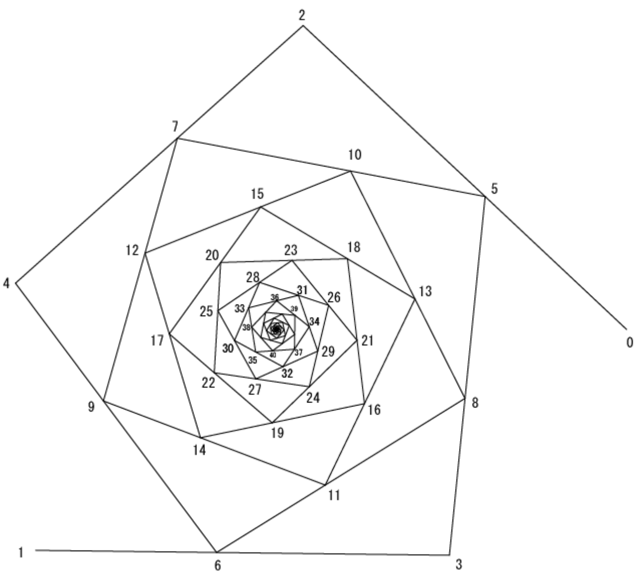


Figure 1: *Fibonacci Tornado.* $[\tau, 3, 5]$

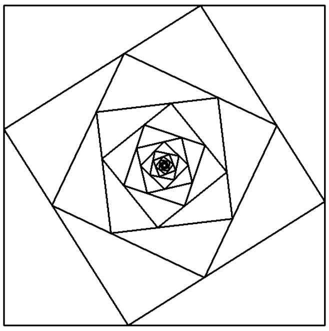


Figure 2 : *A Non-Fibonacci Tornado.*

$$R = C_0 + \cfrac{1}{C_1 + \cfrac{1}{C_2 + \cfrac{1}{C_3 + \cfrac{1}{\ddots\; C_n + \cfrac{1}{\ddots}}}}}$$

p_0/q_0	0th	convergent
p_1/q_1	1st	convergent
p_2/q_2	2nd	convergent
p_3/q_3	3rd	convergent
p_n/q_n	n-th	convergent

Figure 3 : *Continued Fraction and Convergents*

Real Tornado

A generic real number R also generates a family of tornadoes. As is well-known (see [1]), the continued fraction expansion of R as in the Figure 3 is defined by $R = C_0 + \varepsilon_0$, $0 \le \varepsilon_0 < 1$, and $1/\varepsilon_n = C_{n+1} + \varepsilon_{n+1}$, $0 \le \varepsilon_{n+1} < 1$ for $n \ge 0$, where C_n are called the partial denominators. If R is rational, it is related to the Euclidean algorithm and stops when $\varepsilon_n = 0$. The n-th convergent p_n / q_n is defined by $p_0 = C_0$, $q_0 = 1$, $p_1 = C_1 p_0 + 1$, $q_1 = C_1$, and $p_{n+1} = C_{n+1} p_n + p_{n-1}$, $q_{n+1} = C_{n+1} q_n + q_{n-1}$ for $n > 0$. It is known that p_n / q_n are the best approximations of R, where

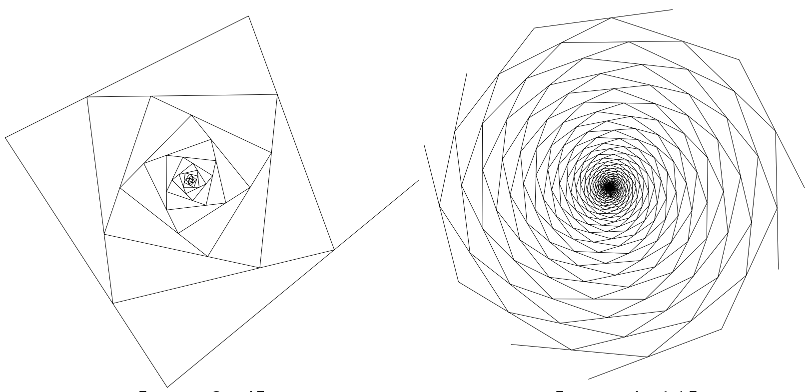
$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < R < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}, \text{ and } \left|\frac{p_n}{q_n} - R\right| > \left|\frac{p_{n+1}}{q_{n+1}} - R\right| \text{ for } n \ge 0 .$$

For example, the convergents of $R = \sqrt{3}$ are 1/1, 2/1, 5/3, 7/4, 19/11, 26/15, 71/41, ... The denominators q_n and q_{n+1} are coprime.

Choose any pair of consecutive convergents p_n / q_n and p_{n+1} / q_{n+1}, and denote by $q = q_n$ and $q' = q_{n+1}$. Define the step size by $k = q' - q$. Then there exists a unique $0 < s < 1$ such that under the Basic Operation the vertex $V(j+q+q')$ lands on the segment $\overline{V(j+q)V(j+q')}$ and we obtain a spiral pattern named as the tornado $[R, q, q']$, consisting of similar triangles $T_j = \Delta V(j)V(j+q)V(j+q')$ for $j \ge 0$. Figure 4 presents the tornadoes $[R, q, q'] = [\sqrt{3}, 3, 4]$ and $[\sqrt{3}, 4, 11]$.

The basic idea of the Real Tornado was originally published in Japanese in [3]. Here we show how to find a $0 < s < 1$. Denote the length of the three edges of T_j by

$$a(j) = |V(j+q)V(j+q')|,$$
$$b(j) = |V(j)V(j+q)|,$$
$$c(j) = |V(j)V(j+q')|.$$



$[\sqrt{3}, 3, 4]$ $[\sqrt{3}, 4, 11]$

Figure 4 : $\sqrt{3}$ *Tornado.*

By Figure 5 we can see that

$$
\begin{aligned}
&a(j)\\
&=|V(j+q)V(j+q')|\\
&=|V(j+q)V(j+q+q')|+|V(j+q+q')V(j+q')|\\
&=s^{q}|V(j)V(j+q')|+s^{q'}|V(j)V(j+q)|\\
&=s^{q}c(j)+s^{q+k}b(j).
\end{aligned}
$$

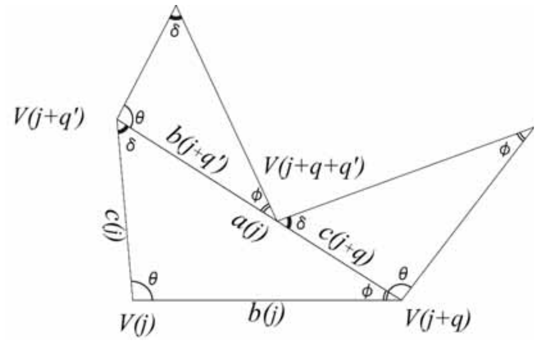


Figure 5 : *Principle.*

The three angles of T_j are

$$\phi = 2\pi Rq' = 2\pi R(q+k)\,,\ \delta = -2\pi Rq\,, \text{ and } \theta = 2\pi Rk$$

or

$$\phi = -2\pi Rq' = -2\pi R(q+k)\,,\ \delta = 2\pi Rq\,, \text{ and } \theta = -2\pi Rk\,,$$

where the signs are chosen to satisfy that $\sin\phi, \sin\delta$ and $\sin\theta$ are all positive. The law of sines is expressed by

$$\frac{a(j)}{\sin\theta} = \frac{b(j)}{\sin\delta} = \frac{c(j)}{\sin\phi},$$

and we obtain the equation

$$s^{q'}\sin(2\pi Rq) - s^{q}\sin(2\pi Rq') + \sin(2\pi Rk) = 0\,.$$

It is easy to see that this equation has a unique solution $0 < s < 1$.

Additional Results

Conversely, we can also prove that any possible tornado $[R, q, q']$ with q, q' positive is related to the continued fraction expansion of R.

Theorem: Let R be a real number and q, q' positive integers. There exists a tornado $[R, q, q']$ if and only if R has a convergent $\frac{p_n}{q_n}$ and an (intermediate) convergent $\frac{cp_n + p_{n-1}}{cq_n + q_{n-1}}, 0 < c \le C_{n+1}$, where we denote by $p = p_n, q = q_n, p' = cp_n + p_{n+1}$ and $q' = cq_n + q_{n+1}$, such that

(1) R is distinct from $\frac{p}{q}$ and $\frac{p'}{q'}$, that is, $\frac{p}{q} < R < \frac{p'}{q'}$ or $\frac{p'}{q'} < R < \frac{p}{q}$, and

(2) $|\{qR\} - \{q'R\}| > 1/2$, where $0 \le \{x\} = x - [x] < 1$ denotes the fractional part.

See [4] for the proof and further discussions. Note that the golden ratio τ is a special irrational number which has no intermediate convergents.

Acknowledgements

The authors would like to thank the reviewers for their helpful comments and suggestions. They suggested to consider the equation $z^{q+k} = \alpha z^{k} + (1-\alpha)$ with $0 < \alpha < 1$ given, where q and k are relatively prime. By experiments, they claim that the tornado $[R, q, q+k]$ is obtained by using the root

$z = se^{2\pi iR} \neq 1$ of the largest magnitude. Note that in our setting above, the ratio α tends to 0 or 1 as R approaches to p/q or p'/q' respectively.

References

[1] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fifth edition, Oxford, 1979.
[2] Akio Hizume, *Real Tornado*, MANIFOLD #17, pp. 8-11. 2008. (in Japanese)
[3] Akio Hizume, *Fibonacci Tornado*, Bridges Proceedings, pp. 485-486. 2008.
[4] Akio Hizume and Yoshikazu Yamagishi, *Monohedral similarity tilings*, in preparation.

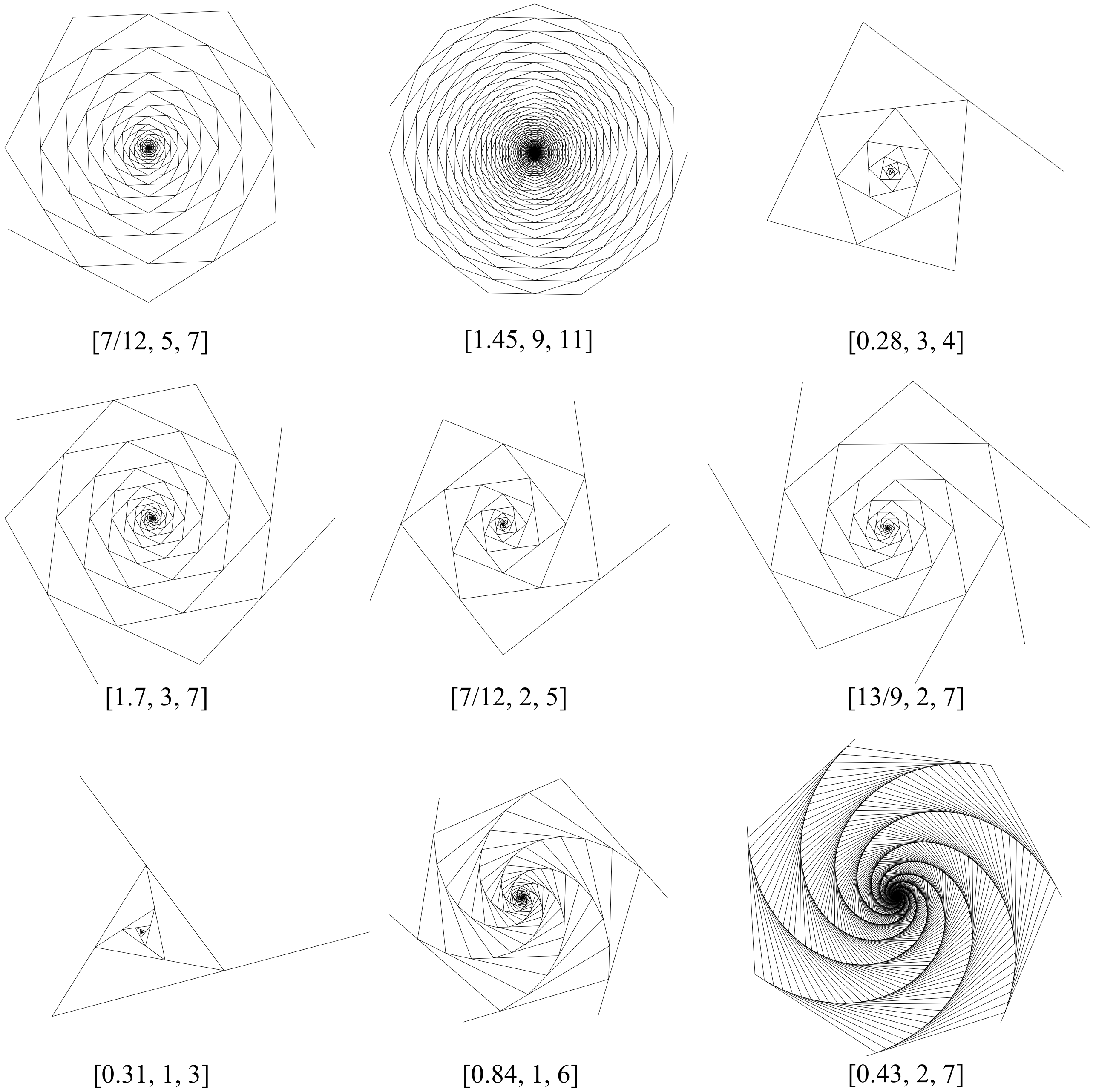


Figure 6 : *Real Tornado Samples.*