Real Number Music
Scale and Tone Based on Continued Fraction

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0. Two Aspects of Simplicity

People tend to think of things as simple. Such an orientation of mind, of course, enables much accomplishment in the field of scientific discovery, including great mathematical theorems and laws of the physical world. On occasion, though, this urge toward simplicity yields not insights but superstitions. I regard these as two aspects of simplicity.

The "Pythagorean Comma" has been a tough issue for musical theorists. I think beliefs about this conjecture come from a secular belief, that 2^(7/12) must equal 3/2, and that the octave is essential for harmony.

The Equal Temperament gave a solution at the sacrifice of perfect consonance. This conventional idea of harmonics, however, has been creating discord between pianists and violinists.

There are some people reclaiming the Natural Temperament, however, with whom I would not agree. When I listened to a contemporary piece set in the Natural Temperament, it sounded quite colorless and dead. I think that modern people who have heard rich expressions derived from the Equal Temperament will not give it up. They will hardly be pleased with the Natural Temperament any more.

This paper proposes a new way to integrate scale and harmony.

1. Preliminary Knowledge

Fibonacci Kecak [Hizume 1995] is a polyrhythmic representation of a recursive structure derived from the Golden Mean, which is an irrational and real number. I extended this derivation to include any real number, using a continued fraction with recursion, in the last paper titled Real Kecak System [Hizume 2001]. Self-similar pulsing that derives from irrational numbers is comfortable to listen to, I believe.

In other treatises, I have adopted the same formula of the Fibonacci Lattice toward tone and scale [Hizume 1990, 1994]. I produced a Golden Mean tone and scale using a synthesizer. The tone was a metallic sound and the scale evoked transcendental feelings.

In this article, I will present a form of tone and scale corresponding to any real number chosen at will.

What I explore is a general formula using minimum parameters and producing unique determinacy.

Furthermore, it should be consistent with and relativize existing musical forms, including those based on the Golden Mean tone and scale.

It is well known that a real number can be expanded into a continued fraction according to the Euclidean Algorithm, as follows:

$R = C_0 + \frac{1}{C_1 + \frac{1}{C_2 + \frac{1}{C_3 + \frac{1}{\ddots\ C_k + \frac{1}{\ddots}}}}}$

q_0/p_0 null convergent
q_1/p_1 1st convergent
q_2/p_2 2nd convergent
q_3/p_3 3rd convergent
q_k/p_k k-th convergent

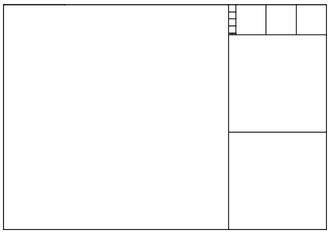


Figure 1: *Division of 157/225 rectangle*

Let a continued fraction be indicated as [C_0, C_1, C_2, C_3, ..., C_k, ...].

Then a relation between the continued fraction and the algorithm can be illustrated as a rectangle divided into squares, as shown in Figure 1. This example is the case of [0, 1, 2, 3, 4, 5], that is 157/225.

Let the rectangle whose ratio of vertical to horizontal is q_k : p_k be divided by C_0 number of squares whose edge is p_k. Only a small rectangle is left unfilled by squares. Looking at the small rectangle, you can see that the major edge is p_k and the minor edge is qk-C_0*p_k.

Now let the leftover rectangle be divided by C_1 number of squares. Except in the case of dividing out, a new, smaller rectangle is left out of the squares.

And so on....

I found the above illustrative technique independently; however, it was already well known.

This schema can adequately define a tone and scale of the Golden Mean, about which I have already published [Hizume 1994], though it could not be extended to include all real numbers. As a result, my early attempts failed to yield a good result, lacking of utilization of recursiveness of the convergent. An audio demonstration is therefore not comfortable to hear.

2. Real Number Music

Subsequently, I found a better way to accurately construct a format of continued fractions that will generate scale and tone.

2. 1. Principal There are two essential parameters to input:

One real number R (0 < R < 1), which-determines scale

Another real number F (0 < F), which-determines the frequency at which the scale will repeat (e.g. any conventional scale is always F=2)

Let us describe a real number as a convergent sequence [0, C1, C2, C3, …, Ck, …]. In this paper, C_0 is always 0, because 0 < R < 1. R and F don't have to relate to each other.

Now I will demonstrate an example of configuring a scale determined by the k-th convergent of R: q_k/p_k. The fraction divides an interval of 1 : F frequency logarithmically into equal p_k partitions. By picking up q_k number of tonal objects, it sets a scale.

It also uniquely determines a spectral arrangement of the tone on the scale.

Consider our earlier example of the rational number 157/225. The continued fraction that produces this number halts at the 5th convergent as [0, 1, 2, 3, 4, 5]. This tonal scale and its spectral arrangement are shown in Figure 2. The horizontal coordinate of the bar graph is logarithmic. We set the left coordinate at 1 and the right end value at F. The arrangement of scale and spectrum repeats every F ratio.

Suppose a lattice divides an interval of 1 : F into 225 partitions, according to the denominator of the input number.

Out of the 225 partitions, sort 43 parts (according to the 4th convergent denominator) 5 times (by the 5th term). There remain 10 parts, which is the denominator of the 3rd convergent.

Out of the 43 partitions, sort 10 parts (according to the 3rd convergent denominator) 4 times (by the 4th term). There remain 3 parts, which is the denominator of the 2nd convergent.

Out of the 10 partitions, sort 3 parts (according to the 2nd convergent denominator) 3 times (by the 3rd term). There remains 1 part, which is the denominator of the 1st convergent.

Out of the 3 partitions, sort 1 part (according to the 1st convergent denominator) 2 times (by the 2nd term). There remains 1 part, which is the denominator of the null convergent.

The elegant relation shown above is the principle.

These operations, which I call the Inverse-Euclidean Algorithm, can be illustrated as a right triangle recursively divided by a set of smaller right triangles.

It might be an application to approximate a straight line using computer graphics.

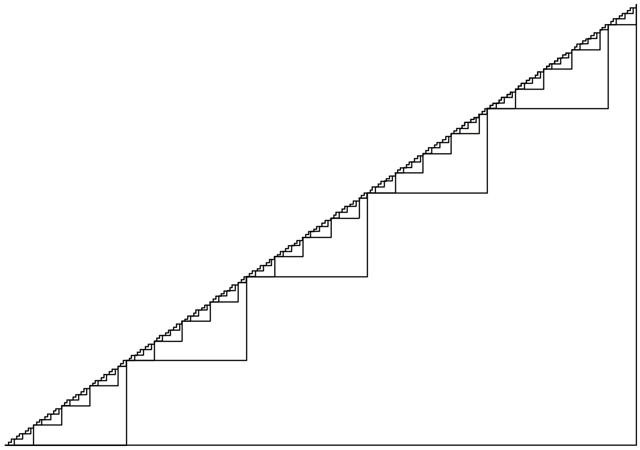
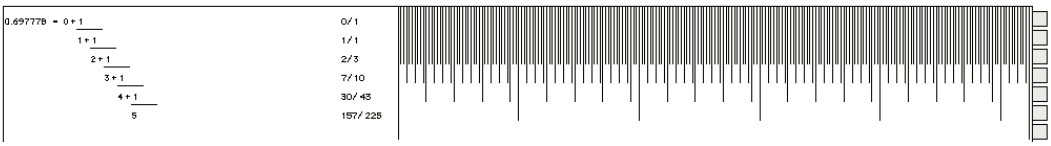


Figure 2:
Recursive division of 157/225



2.2. Scale Regarding Figure 2 as information about scale, an arrangement of divisions can be translated any number of times to a higher frequency band than F, and a lower band than 1.

Especially, given F as infinite and R as an irrational number, such a continued fraction, having no limit in its expansion, results in an everlasting aperiodic scale.

The length of the vertical bars represents a rank of scale. If you pick only those bars that exceed a certain length, then you gain an abridged version of the scale.

The maximum division of scale is just an equal pk tempered scale.

2.3. Spectral Map Regarding Figure 2 as a spectral map, the left periphery f, fundamental frequency, and the right one f*F, correspond to the pk-th partial tone. In an interval of frequency ratio 1 : F, in fact, there are pk number of partial tones. Then grant ranks of amplitude according to each length of a bar graph, and proportionate degree of ranks to numerator of the convergent. An arrangement of division in an interval of f : f*F can be repeated any number of times to higher frequency band than f*F.

Note, however, that the length of these bars implies not amplitude but rank. To avoid damaging your ears, you must reduce the amplitude as the frequency rises.

I have therefore set the amplitude of each spectrum as a product of a rank of bar graph (that is the convergent numerator of the rank) and an inverse value of the frequency.

For example, in the case of Figure 2, where the length of the bars at the right and left peripheries is 157, I would set the leftmost amplitude to 157/F and set the rightmost amplitude to 157/1
In such a way, any real number produces a raw and distinctive tone.

This spectral map reminds me of the hairy cells of Corti's organ in the inner ear. Our ears always generate an infinite number of sine waves and hear them. We distinguish tones according to which hairy cell (that is corresponding to a spectrum) is stimulated.

According to the principle above, the conventional octave is redefined as R=7/12 and F=2/1.

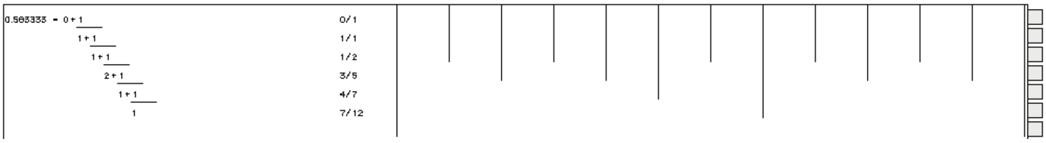


Figure 3: *Arrangement of 7/12 tonal scale and spectrum*

2.4. A New Perspective on Harmonics Scale and tone thus accurately reflect a recursive format of a continued fraction derived from a real number.

Therefore, you can alter your thoughts about harmonics dating from Pythagorean days.

The conventional definition of consonance has been based on valuating ratio of frequency. That is, the simpler integral a ratio is, the higher it resolves. Now you can discard such a definition, and I offer you a new option, based on rating how many spectral terms two tones have in common.

Since tonal scale and spectral arrangement form a pattern, two tones whose ratio on fundamental is 1 : F have the same arrangement of spectrum. And we feel they are the same tone.

For example, in the case of the conventional scale determined by 7/12, two tones whose ratio on fundamental is 1 : 2^n (where n is a natural number) resolve best. Other cases resolve according to a recursive format of a real number.

Imagine two copies of an arrangement of spectra, for example, those shown in Figure 2, 3, or 4, and then overlap and see through them variously, to focus on apparent patterns.

From this viewpoint where scale and tone are thus integrated perfectly, the Pythagorean Comma is out of the question, and you have an entirely new understanding of harmony.

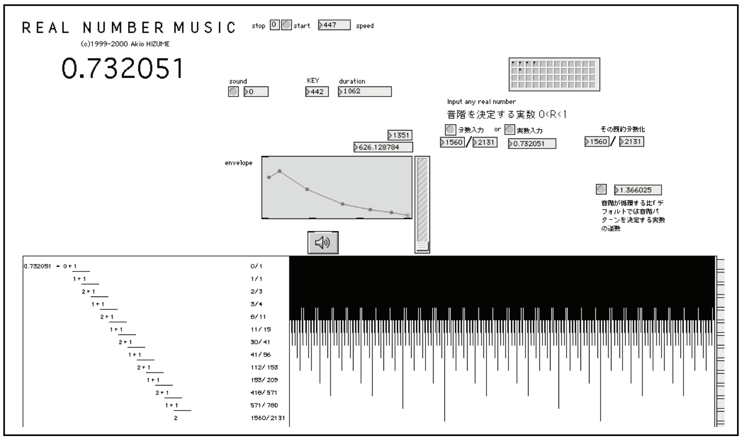


Figure 4: *A sample image of an active window of a program made of MAX+MSP. It explains a tonal scale and spectrum generated by* $\sqrt{3}$*-1. It extends a real number into a continued fraction and sound a distinctive tonal scale and tone color according to the number selected. You can add any envelope effect to the tone at will.*

Acknowledgement:

I would like to thank the Shoji Yotsutani Lab. of Ryukoku University who supported my musical research.

This paper is a translation of the Japanese paper which was published in MANIFOLD #03, 2001. Mr. Peter Engel and Mr. Masahiro Hirayama gave me suggestions for my English expression.